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On the critical fluctuations in superconductors

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Abstract. The field-theoretical gauge model for a superconductor, generalized to an $n/2$-component complex order parameter is considered. The question of the order of the phase transition occurring in this model is discussed. Previous renormalization-group calculations suggest that in such a model a fluctuation-induced first-order phase transition occurs. We re-examine previously obtained expressions for the renormalization-group functions in the two-loop approximation in three dimensions. Special attention is being paid to the fact that the corresponding series might be the asymptotic ones and therefore have zero radius of convergence. We discuss the possible ways of analytical continuation of the series obtained. Comparing results obtained by 'direct' calculations with those obtained by Padé analysis and the Padé–Borel resummation technique, the conjecture is made that in the model under consideration there still exists a possibility for a second-order phase transition.

1. Introduction

The smallness of the correlation length ξ_0 in high-T_c superconductors leads to an increase of the temperature region near T_c where fluctuation effects [1] might be observed. The general belief in this field is that the phase transition is weakly of first order [2] (for another model see [3]). This fluctuation-induced first-order transition could only be seen within such a small temperature region round T_c that it is unobservable in experiment. In low-T_c superconductors with a large correlation length, ξ_0 mean-field behaviour is observed, since fluctuations are relevant only unobservable near T_c as the Ginzburg criterion states. Support for this picture has been given in liquid crystals, where the same picture applies [4].

In high-T_c superconductors in several experiments critical effects have been observed in the specific heat [5, 6]. They have been analysed with scaling exponents related to the fixed point in the uncharged model [7]. In the presence of a magnetic field, where some of the experiments have been performed, the situation is more complicated, however, and the question of the order of the transition remains [8].

The theoretical model which is used to describe the relevant critical behaviour is the usual ϕ^4-model coupled to a gauge field [2]. The minimal coupling to the gauge field makes the model (i) different from the model describing the superfluid transition in ^{4}He and (ii) leads, as is the general belief, to a first-order transition instead of a continuous transition as in the uncharged system of ^{4}He. The arguments for a first-order transition in this model are based on a runaway solution instead of a fixed point in the space of the static parameters found in *one-loop order*. In [9] we calculated the two-loop flow equations for the static parameters and indicated that a stable fixed point might be possible, consequently a second-order transition would appear.

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An attractive feature of the flow found in [9] was that it discriminated between type-I and type-II superconductors, depending on the initial (background) values of the couplings. For small values of the ratio (coupling to the gauge field)/(fourth-order coupling) (appropriate for type-II superconductors) the flow comes very near to the fixed point of the uncharged model but ends in the new superconducting fixed point. For large values of the ratio (type-I superconductors) the flow runs away. For values of the ratio in between, the critical behaviour might be influenced by a second (unstable) superconducting fixed point with scaling exponents quite different from the uncharged model.

Monte Carlo simulations of the three-dimensional lattice superconductor model [10, 11] indicate a second-order phase transition moving from the type-I superconductor region to the type-II superconductor regime, although the possibility always exists of a very small first-order transition in the type-II region. More recently, from a self-consistent theory of the normal-to-superconductor transition it was concluded that the first-order transition might be an artefact of the breakdown of the ε-expansion and a non-trivial critical point was predicted [12].

In this paper we therefore reconsider the two-loop equations and the resulting flow and calculate effective exponents, given by the zeta function, which have also been calculated in two-loop order in [9]. The main point here is that we take account of the fact that the loop expansion is only an asymptotic one. We apply resummation techniques to the beta functions of the flow as well as to the zeta functions. In this way we find several fixed points with new scaling exponents and a rich crossover behaviour.

The structure of this paper is as follows. In section 2 we describe the model we are interested in, give the expressions for the renormalization-group functions in the two-loop approximation and describe the results obtained on their basis without applying a resummation procedure. Section 3 is devoted to the study of the β-functions which are the functions of two variables and corresponding flows on the base of the Padé–Borel resummation technique and Padé approximants for the appropriate resolvent series. In section 4 we calculate the asymptotic and effective values for the critical exponents. The results are discussed in section 5.

2. General considerations

2.1. The model

Starting from the Landau–Ginsburg free-energy functional $F(\Psi, \boldsymbol{A})$ for a generalized superconductor in d dimensions with the vector potential $\boldsymbol{A}$ and the order parameter Ψ consisting of $n/2$ complex components one can describe the fluctuation effects by an Abelian Higgs model with the gauge-invariant Hamiltonian [2]:

$$H = \int \mathrm{d}^d x \, \left\{ \frac{1}{2} t_0 |\Psi_0|^2 + \frac{1}{2} |(\nabla - \mathrm{i} e_0 \boldsymbol{A}_0)\Psi_0|^2 + \frac{u_0}{4!} |\Psi_0|^4 + \frac{1}{2} (\nabla \times \boldsymbol{A}_0)^2 \right\} \tag{1}$$

depending on the bare parameters t_0, e_0, u_0. The parameter t_0 changes its sign at some temperature, the rest of the parameters being considered as temperature-independent. For the coupling constant $e_0 = 0$ no magnetic fluctuations are induced and the model reduces to the usual field theory describing a second-order phase transition and corresponding to the particular case $n = 2$ to the superfluid transition in ^{4}He.

Recent two-loop results [9] for the renormalization-group functions corresponding to (1) were obtained on the basis of dimensional regularization and minimal subtraction scheme [13] (for calculations at $d = 3$ see [14, 15]). The flow equations for the renormalized couplings $u,\ f\ (f = e^2)$ are

$$l\frac{\mathrm{d}f}{\mathrm{d}l} = \beta_f \tag{2}$$

$$l\frac{\mathrm{d}u}{\mathrm{d}l} = \beta_u \tag{3}$$

where l is the flow parameter and the expressions for β-functions in the two-loop approximation read [9]

$$\beta_f = -\varepsilon f + \frac{n}{6}f^2 + nf^3 \tag{4}$$

$$\beta_u = -\varepsilon u + \frac{n+8}{6}u^2 - \frac{3n+14}{12}u^3 - 6uf + 18f^2 + \frac{2n+10}{3}u^2 f + \frac{71n+174}{12}uf^2 - (7n+90)f^3 \tag{5}$$

with $\varepsilon = 4-d$. The previous analysis of the equations of type (4) and (5) either in one-loop [2] or in two-loop order [9] was based on a direct solution of the equation for the fixed point. In the present study we want to attract attention to the fact that the series have zero radius of convergence and that they are known to be asymptotic at best. Therefore some additional mathematical methods have to be applied in order to obtain reliable information on their basis. In fact, the asymptotic nature of the series for the renormalization-group functions has been proved only in the case of the ϕ^4 model containing one coupling of $O(n)$-symmetry (n-vector model) where the high-order asymptotics for these series is known [16–18] in analytical form. These results gave the possibility of obtaining precise values of the critical exponents for the n-vector model by resummation of the corresponding series for the renormalization-group functions (see e.g. [14, 19, 20]). For the ‘charged’ model we are considering here there is, to our knowledge, no available information similar to that obtained in [16–18] for the ‘uncharged’ case ($f = 0$). Nevertheless, in the case of a model containing several couplings of different symmetry the asymptotic nature of the corresponding series for the renormalization-group functions is rather a general belief than a proven fact. As one such example we mention here the weakly diluted n-vector model, described by a Hamiltonian containing two fourth-order terms of different symmetry [21]. The asymptotic nature of the double series for the renormalization-group functions in terms of the coupling constants has not been proven for this model up to now†. However, an appropriate resummation technique (applied *as if* these series are the asymptotic ones) enables one to obtain accurate values for critical exponents in three dimensions [22–25] and to describe (in the $n = 1$ case) the experimentally observed crossover to a new type of critical behaviour caused by weak dilution [26, 27]. These results are also confirmed by Monte Carlo [28, 29] and Monte Carlo renormalization-group [30] calculations. We therefore do not hesitate to use these methods in our case.

2.2. ε-expansion results

Keeping all the above-mentioned problems in mind, let us have a closer look at the series (4) and (5). We start by recalling results of a ε^2-expansion for the β-functions [2, 9]. In second order in ε one obtains three fixed points: the Gaussian ($u^{*\mathrm{G}} = f^{*\mathrm{G}} = 0$), the ‘uncharged’ ($u^{*\mathrm{U}} \neq 0,\ f^{*\mathrm{U}} = 0$) and the ‘charged’ ($u^{*\mathrm{C}} \neq 0,\ f^{*\mathrm{C}} \neq 0$), to be denoted as G, U, C. The

† Only for the case when one of the couplings is equal to zero does one obtain a series which is proved to be the asymptotic one.

expressions for them read

$$\mathrm{G}: \; u^{*\mathrm{G}} = 0 \qquad f^{*\mathrm{G}} = 0 \tag{6}$$

$$\mathrm{U}: \; u^{*\mathrm{U}} = u_1^U \varepsilon + u_2^U \varepsilon^2 \qquad f^{*\mathrm{U}} = 0 \tag{7}$$

$$\mathrm{C}: \; u^{*\mathrm{C}} = u_1^C \varepsilon + u_2^C \varepsilon^2 \qquad f^{*\mathrm{C}} = f_1^C \varepsilon + f_2^C \varepsilon^2 \tag{8}$$

where

$$u_1^U = \frac{6}{n+8} \qquad u_2^U = \frac{18(3n+14)}{(n+8)^3}$$

$$u_1^C = \frac{3(n+36) + (n^2 - 360n - 2160)^{1/2}}{3n(n+8)} \qquad u_2^C = \frac{a_2}{a_1}$$

$$f_1^C = \frac{6}{n} \qquad f_2^C = -\left(\frac{6}{n}\right)^3 n$$

$$a_1 = 1 + \frac{n+8}{3} u_1^C - \frac{36}{n}$$

$$a_2 = \frac{3n+14}{12} \left(u_1^C\right)^3 - 6nu_1^C \left(\frac{6}{n}\right)^3 + 36n \left(\frac{6}{n}\right)^4 - \frac{(n+5)4}{n} (u_1^C)^3$$
$$- \frac{3(71n+174)}{n^2} u_1^C + \left(\frac{6}{n}\right)^3 (7n+90) .$$

Almost all physical results concerning the phase transition described by the field theory (1) were based on the information given by (6)–(8). The main ones read:

(i) the fixed point U is unstable in the f-direction at $d < 4$ with the stability exponent

$$\lambda_f(u = u^{*\mathrm{U}}, f = f^{*\mathrm{U}} = 0) = \left.\frac{\partial \beta_f}{\partial f}\right|_U = -\varepsilon ;$$

(ii) the fixed point C appears to be complex for $n < n_c = 365.9$ [2] already in one-loop order. The stability exponent given by

$$\lambda_u(u = u^{*\mathrm{C}}, f = f^{*\mathrm{C}}) = \left.\frac{\partial \beta_u}{\partial u}\right|_C$$

and in the two-loop order reads

$$\lambda_u = -\varepsilon s \qquad s = \left[\left(1 + \frac{36}{n}\right)^2 - \frac{432(n+8)}{n^2}\right]^{1/2}$$

leading to an oscillatory flow in u in one-loop order below n_c with the solution [9, 37]:

$$f(l) = \frac{6fl^{-\varepsilon}}{6 + n\varepsilon f(l^{-\varepsilon} - 1)} \tag{9}$$

$$u(l) = f(l) \frac{n}{2(n+8)} \left\{ s \tan \left[\frac{s}{2} \ln(f(l) f^{-1} l^{\varepsilon}) \right.\right.$$
$$\left.\left. + \arctan \left(\frac{2(n+8)}{sn} \frac{u}{f} + \frac{n+36}{ns} \right) \right] - \frac{n+36}{n} \right\} \tag{10}$$

where f and u are the initial parameters at $l = 1$;

(iii) from the condition of positiveness of the fixed-point coordinate f^* ($f = e^2$) it follows that at $\varepsilon = 1$ $n > 36$.

Finally, one concludes that for the 'superconductor' case $n = 2$ of most physical interest no stable fixed point exists and therefore the observed phase transition is of first order.

2.3. Resummation

Nevertheless, one should note that such a straightforward interpretation of the ε-expansion data is questionable and a method of analysis of the series for β-functions (4), (5) avoiding strict ε-expansion and exploiting information about the accurate solution for the pure model case at $d = 3$ was proposed [9]. Also from the comparison of ε-expansion values for f^* (giving a positive value of f^* only for $n > 36$) with the value of f^* obtained without ε-expansion (remaining positive for all n) the conjecture was made that the lower boundary for n resulting in a negative f^* might be an artefact of the expansion procedure.

As is well known nowadays the appropriate resummation technique applied in the theory of critical phenomena to the asymptotic series for the renormalization-group functions enables one to obtain extremely accurate values of the critical exponents. Two main ways of resummation are commonly used: (i) resummation based on the conformal mapping technique and (ii) Padé–Borel resummation. Case (i) is based on the conformal transformation, which maps a part of the domain of analyticity containing the real-positive axis onto a circle centred at the origin and the asymptotic expansion for a certain function is thus re-written in the form of a new series (see [20] for a discussion). However, this resummation is based on the knowledge of subtle details of the asymptotics (location of the pole, high-order behaviour) which are not available in our case. In the absence of any knowledge about the singularities of the series the most appropriate method which can be used to perform the analytical continuation is the Padé approximation resulting in Padé–Borel resummation techniques (ii) (see e.g. [19]). In the following we are going to apply this method to the special case of $f = 0$, so let us concentrate on it in detail.

Starting from the Taylor series for the function $f(u)$:

$$f(u) = \sum_{j\geqslant 0} c_j u^j \tag{11}$$

one constructs the Borel transform

$$F(ut) = \sum_{j\geqslant 0} \frac{c_j}{j!}(ut)^j \,. \tag{12}$$

Then one represents (12) in the form of the Padé approximant $F^{\text{Padé}}(ut)$ (in the subsequent analysis, proceeding in the two-loop approximation we will use the [1/1] Padé approximant) and the resummed function is given by

$$f^{\text{Res}}(u) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} F^{\text{Padé}}(ut) \,. \tag{13}$$

However, the technique described above, generally speaking, could be applied only to an alternating series like

$$f(u) = \sum_{j\geqslant 0} (-1)^j c_j u^j \qquad c_j > 0 \,. \tag{14}$$

The problem is that fractions of the type c_j/c_{j+1} enter the denominator of the Padé approximant and a pole in the integral representation of the resummed function could appear if the series is not an alternating one†. In the next section we will show how this procedure works in the case when only one coupling, u, is present.

† In this case the principal value of the integral (13) could be taken, but generally speaking it is preferable to avoid such situations (see also [19]).

3. Fixed points and flows in three dimensions

We will proceed here by considering the flow equations (2) and (3) directly at $d = 3$. Let us look for the solutions of the fixed point equations at $d = 3$ paying attention to the possible asymptotic nature of the corresponding series (4) and (5).

3.1. The uncharged fixed point U

Substituting the value $f^* = 0$ into (5) one obtains the following expression for the function $\beta_u^{\mathrm{U}} \equiv \beta_u(u, f^* = 0)$:

$$\beta_u^{\mathrm{U}} = -u + \frac{n+8}{6}u^2 - \frac{3n+14}{12}u^3 . \tag{15}$$

Solving this polynomial for the fixed point one obtains for the non-trivial $u^* > 0$:

$$u^{*\mathrm{U}} = \frac{n+8}{3n+14} + \frac{\sqrt{n^2 - 20n - 104}}{3n+14} \tag{16}$$

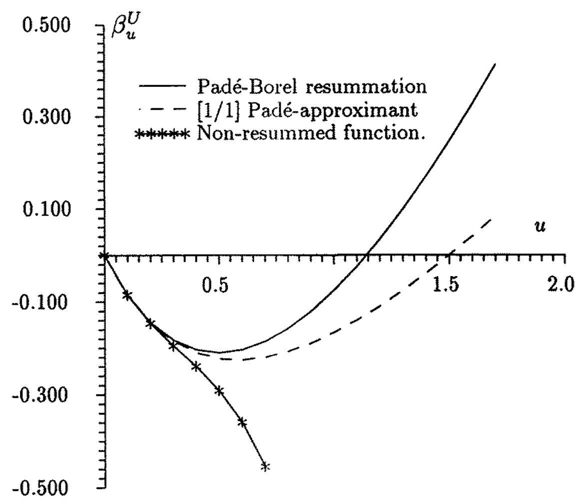

Figure 1. β_u-function of the uncharged model β_u^{U} at $d = 3$, $n = 2$.

and immediately the 'condition of existence of non-trivial solution $u^{*\mathrm{U}}$' follows qualitatively, very similar to those appearing in the strict ε-expansion technique (see [2, 9] and (7) of the present paper as well): the solution exists only for certain values of $n > n_c = 24.3$! From figure 1 one can see that the function β_u^{U} (equation (15)) does not intersect the u-axis at any non-zero value of u for $n = 2$. In the $O(n)$-symmetric ϕ^4-theory at $d = 3$ this situation is well known (see e.g. [31, 32]): the β-function calculated directly at $d = 3$ does not possess a stable zero for realistic values of n, nevertheless, in three-loop order the presence of the stable fixed point is restored. To avoid this artefact appearing in the two-loop calculation one can either resum the series for the β-function or construct the appropriate Padé approximant† in order to perform the analytical continuation of (15) out of the domain of convergence

† The last possibility was chosen by Parisi [31] in order to restore the presence of a stable solution for the fixed point in the two-loop approximation.

(which is equal to zero for the series in the right-hand side of (15). Let us try both ways. Representing (15) in the form of the [1/1] Padé approximant:

$$\beta_u^{\mathrm{U,Padé}} = u\,\frac{-1 + A_u u}{1 + B_u u} \tag{17}$$

one obtains

$$A_u = \frac{n^2 + 7n + 22}{6(n+8)} \qquad B_u = \frac{3n+14}{2(n+8)} \tag{18}$$

and, solving the equation for the fixed point

$$\beta_u^{\mathrm{U,Padé}}(u^{*\mathrm{P,Padé}}) = 0 \tag{19}$$

one obtains

$$u^{*\mathrm{U,Padé}} = \frac{6(n+8)}{n^2 + 7n + 22}\,. \tag{20}$$

So we obtained a qualitatively different situation. The behaviour of $\beta_u^{\mathrm{U,Padé}}(u)$ for $n = 2$ is shown in figure 1 by the broken curve. If one is interested in more accurate values of u^* some resummation has to be applied. Choosing the Padé–Borel resummation technique† and following the scheme (11)–(13) one obtains for the resummed function $\beta_u^{\mathrm{U,Res}}$:

$$\beta_u^{\mathrm{U,Res}} = u\left[2(1 - A_u/B_u)\left(1 - E\left(\frac{2}{uB_u}\right)\right) - 1\right] \tag{21}$$

the coefficients A_u, B_u are given by (18), $E(x) = x\mathrm{e}^x E_1(x)$, where the function

$$E_1(x) = \mathrm{e}^{-x}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}(x+t)^{-1}$$

is connected with the exponential integral by the relation [33]:

$$E_1(x \pm \mathrm{i}0) = -\mathrm{Ei}(-x) \mp \mathrm{i}\pi\,.$$

The behaviour of the function $\beta_u^{\mathrm{U,Res}}(u)$ is shown in figure 1 by the full curve. And the fixed point coordinate $u^{*\mathrm{U,Res}}$ is obtained by solving the non-linear equation

$$\beta_u^{\mathrm{U,Res}}(u^{*\mathrm{U,Res}}) = 0\,. \tag{22}$$

Coordinates of the fixed point $u^{*\mathrm{U}}$ obtained on the basis of the Padé approximation and Padé–Borel resummation ($u^{*\mathrm{U,Padé}}$, $u^{*\mathrm{U,Res}}$) for different n are given in table 1.

Table 1. Fixed-point U coordinate $u^{*\mathrm{U}}$ as a function of n. $u^{*\mathrm{U,Padé}}$: obtained on the basis of the [1/1] Padé approximant; $u^{*\mathrm{U,Res}}$: obtained by Padé–Borel resummation.

n	$u^{*\mathrm{U,Padé}}$	$u^{*\mathrm{U,Res}}$
1	1.800	1.315
2	1.500	1.142
3	1.269	1.002
4	1.091	0.888
5	0.951	0.794
6	0.840	0.717
7	0.750	0.652
8	0.676	0.597

† The series in (15) appears to be an alternating one and this scheme can be applied without any difficulties.

Table 2. Fixed-point C coordinate $f^{*\mathrm{C}}$ as a function of n. $f^{*\mathrm{C,Dir}}$: obtained by direct solution of the equation for the fixed point; $f^{*\mathrm{C,Padé}}$: obtained on the basis of [1/1] Padé approximant; $f^{*\mathrm{C},\varepsilon}$: ε-expansion result with linear accuracy in ε; $f^{*\mathrm{C},\varepsilon^2}$: ε-expansion result with square accuracy in ε.

n	$f^{*\mathrm{C,Dir}}$	$f^{*\mathrm{C,Padé}}$	$f^{*\mathrm{C},\varepsilon}$	$f^{*\mathrm{C},\varepsilon^2}$
1	0.920	0.162	6.000	−210.000
2	0.629	0.158	3.000	−51.000
3	0.500	0.154	2.000	−22.000
4	0.424	0.150	1.500	−12.000
5	0.372	0.146	1.200	−7.440
6	0.333	0.143	1.000	−5.000
7	0.304	0.140	0.857	−3.551
8	0.280	0.136	0.750	−2.625

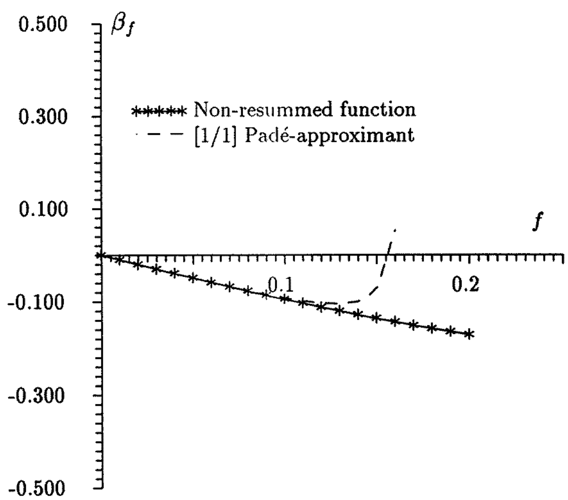

Figure 2. β_f-function at $d = 3, n = 2$.

We conclude from this analysis: in the $d = 3$ theory the Padé approximants (as an analytical continuation of β-functions) qualitatively may change the picture and lead to values of the fixed points comparable to those obtained by the Padé–Borel resummation technique.

3.2. Charged fixed point C

Let us now apply the above considerations to β_f for which the expression at $d = 3$ reads (equation (4)):

$$\beta_f = -f + \frac{n}{6} f^2 + n f^3 \,. \tag{23}$$

The behaviour of β_f as a function of f is shown in figure 2 by asterisks. Note, however, that in this case the function β_f, even without any resummation, possesses a non-trivial zero f^{*M} (its value $f^{*\mathrm{C,Dir}}$ is given in the second column of table 2). Representing (23) in a form of the [1/1] Padé approximant:

$$\beta_f^{\text{Padé}} = f\,\frac{-1 + A_f f}{1 + B_f f} \tag{24}$$

one has for A_f, B_f:

$$A_f = \frac{n+36}{6} \qquad B_f = -6 \tag{25}$$

and, solving the equation for the fixed-point coordinate $f^{*\mathrm{C,Padé}}$:

$$\beta_f^{\text{Padé}}(f^{*\mathrm{C,Padé}}) = 0 \tag{26}$$

one obtains

$$f^{*\mathrm{C,Padé}} = \frac{6}{n+36}\,. \tag{27}$$

The function $\beta_f^{\text{Padé}}(f)$ is shown in figure 2 by the broken curve, the coordinate $f^{*\mathrm{C,Padé}}$ is given in the third column of table 2. But now the series (23) is not alternating and this results in the presence of a pole (at $f = \frac{1}{6}$) in the approximant (24). Therefore equation (24) correctly represents the function $\beta_f(f)$ only for $f < \frac{1}{6}$. Let us note, however, that for all positive values of n a fixed point exists and its coordinate $f^{*\mathrm{M,Padé}}$ lies within the limits $0 < f^{*\mathrm{C,Padé}} < \frac{1}{6}$, where no pole in (24) exists. Comparing this result with those obtained in the previous subsection one can conclude that the representation of β_f in the form of the Padé approximant does not qualitatively change the picture (a solution for $\beta_f(f) = 0$ exists at $d = 3$ even without an analytical continuation) but results in a decrease of the fixed-point coordinate. In contrast to the ε-expansion values (8) there do not exist any border line values of n for the positivity of $f^{*\mathrm{C}}$. Unfortunately, we cannot check this result by means of the Padé–Borel resummation technique: the above-mentioned presence of a pole in the denominator of the Padé approximant makes the corresponding integral representation problematic.

In order to find the u-coordinate of the fixed point C, $u^{*\mathrm{C}}$, we have to deal with a function of two variables, $\beta_u(u, f)$, represented by a rather short series (5). One additional problem arises due to the fact that function $\beta_u(u, f)$ contains generating terms (i.e. $\beta_u(u = 0, f) \neq 0$). In order to perform some kind of analytic continuation of a function of two variables one can use rational approximants of two variables (so-called Canterbury approximants or generalized Chisholm approximants [34, 35]) being the generalization of Padé approximants in the case of several variables. But the presence of generating terms makes this choice rather ambiguous. The most reliable way in such a case seems to be a representation of $\beta_u(u, f)$ in the form of a 'resolvent' series $B(u, f, t)$ [35, 36] introducing an auxiliary variable t, which allows the separation of contributions from different orders of the perturbation theory in the coupling constant. The series for $B(u, f, t)$ then reads

$$B(u, f, t) \equiv \beta_u(ut, ft) = \sum_{j\geqslant 0} b_j t^j \tag{28}$$

with obvious notation for the coefficients b_j. Now one considers (28) as a series in the *single* variable t. This series can be represented in a form of Padé approximant $B^{\text{Padé}}(u, f, t)$ as the analytical continuation of the function $B(u, f, t)$ for a general value of t. In particular, at $t = 1$ the equality holds $B(u, f, t = 1) = \beta_u(u, f)$ and the approximant

$$B^{\text{Padé}}(u, f, t = 1) \equiv \beta_u^{\text{Padé}}(u, f)$$

represents the initial function $\beta_u(u, f)$. In our case the expression for $B(u, f, t)$ reads

$$B(u, f, t) = t(b_1 + b_2 t + b_3 t^2) \tag{29}$$

where

$$b_1 = -u \qquad b_2 = \frac{n+8}{6}u^2 - 6uf + 18f^2$$
$$b_3 = -\frac{3n+14}{12}u^3 + \frac{2n+10}{3}u^2 f + \frac{71n+174}{12}uf^2 - (7n+90)f^3 .$$

Representing the expression in brackets on the right-hand side of (29) in the form of the [1/1] Padé approximant we have

$$B^{\text{Padé}}(u, f, t) = tb_1 \frac{1 + A_{u,f}t}{1 + B_{u,f}t} \tag{30}$$

where

$$A_{u,f} = \frac{b_2}{b_1} - \frac{b_3}{b_2} \qquad B_{u,f} = \frac{-b_3}{b_2} . \tag{31}$$

Let us note here that the function $B(u, f, t)$, as the approximant for the function of two variables $\beta_u(u, f)$, obeys certain projection properties in the single-variable case: substituting $f = 0$ or $u = 0$ into (30) one obtains the [1/1] Padé approximant for $\beta_u^{\text{U}}(u)$ or the [0/1] Padé approximant for $\beta_u(u = 0, f)$. Finally, the expression for $\beta_u(u, f)$ approximated in such a way reads

$$\beta_u^{\text{Padé}}(u, f) = b_1 \frac{1 + A_{u,f}}{1 + B_{u,f}} . \tag{32}$$

Substituting into the equation for the fixed point $\beta_u(u^{*\text{C}}, f^{*\text{C}}) = 0$ the value for the coordinate $f^{*\text{C}} = f^{*\text{C,Padé}}$ (equation (27)) one obtains the non-linear equation for $u^{*\text{C,Padé}}$:

$$\beta_u^{\text{Padé}}(u, f = f^{*\text{C,Padé}}) = 0 . \tag{33}$$

Solving equation (33) with respect to u one obtains the values $u^{*\text{C,Padé}}$ given in table 3. The intersection of the function $\beta_u^{\text{Padé}}(u, f)$ (equation (32)), with the plane $f = f^{*\text{C,Padé}}$ is shown for $n = 2$ in figure 3. The first fixed point (C1) given in the second column of table 3 turns out to be unstable, while the fixed point C2 is stable also for the case $n = 2$ we are mainly interested in.

Table 3. Fixed-point C coordinates $u^{*\text{C,Padé}}$ obtained on the basis of the [1/1] Padé approximant for the 'resolvent' series as a function of n. C1: unstable fixed point; C2: stable fixed point.

	$u^{*\text{C,Padé}}$	
n	C1	C2
1	0.184	3.309
2	0.181	2.457
3	0.179	1.781
4	0.177	1.150
5	0.175	0.473
6	0.175	0.369
7	0.176	0.305
8	0.179	0.256

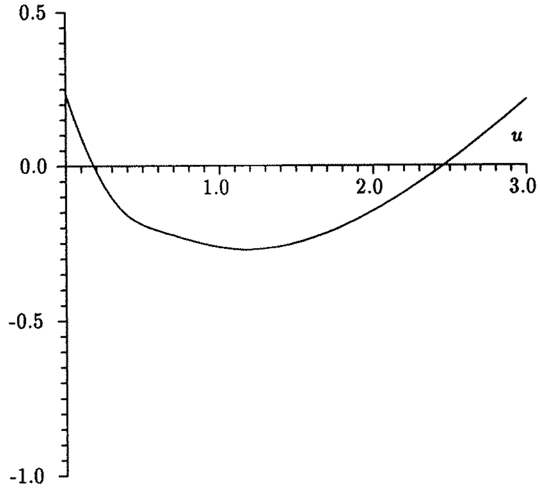

Figure 3. Intersection of the function $\beta_u^{\text{Padé}}(u, f)$ at $d = 3$, $n = 2$ with the plane $f = f^{*\text{C,Padé}}$ in the two-loop approximation.

3.3. Flows

The crossover to the asymptotic critical behaviour is described by the solutions of the flow equations (2), (3) with initial values $u(\ell_0)$ and $f(\ell_0)$ at $\ell = \ell_0$†. Substituting for the β-functions entering the right-hand side of (2), (3) their analytical continuation in the form of the Padé approximants (24), (32) we get the following system of differential equations:

$$l\frac{\mathrm{d}f}{\mathrm{d}l} = f\frac{-1 + A_f f}{1 + B_f f} \tag{34}$$

$$l\frac{\mathrm{d}u}{\mathrm{d}l} = -u\frac{1 + A_{u,f}}{1 + B_{u,f}} \tag{35}$$

where A_f, B_f and $A_{u,f}$, $B_{u,f}$ are given by (25) and (31) correspondingly.

Solving equations (34), (35) numerically one gets the flow diagram shown in figure 4 for the case $n = 2$. The space of couplings is divided into several parts by separatrices (thick curves in figure 4) connecting the fixed points. Besides the Gaussian (G) there exist three fixed points, one corresponding to the uncharged (U) and two others corresponding to the charged (C1, C2) cases. The fixed points G, C1 and U are unstable (full circles in figure 4) and the fixed point C2 is the stable one (shown as a full square in figure 4). Several different flow lines are shown in figure 4. They can be compared with the corresponding flow picture obtained by a direct solution of the flow equations for the two-loop β-functions expressed by the third-order polynomials in the couplings u, f (equations (4), (5)) (see figure 2(a) in [9]). There one can see that no stable fixed point existed and even the fixed point U was absent. Comparing figure 4 and figure 2(b) from [9] one can see how an analytical continuation of the β-functions (4), (5), done only partly in [9] and performed here in the form of Padé approximants, restores the presence of the fixed point U (unstable) and leads to the appearance of a new stable fixed point C2 for the charged model. The coordinates of the fixed points U, C1, C2 are given in the corresponding columns of tables 1–3 and for

† We take $\ell_0 = 1$.

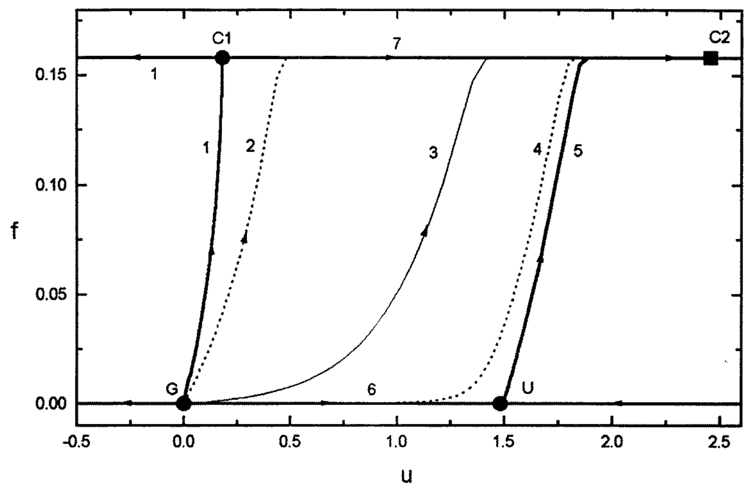

Figure 4. Flow lines for the case $n = 2$, $d = 3$ given by equations (4), (5) (for further description see the text).

$n = 2$ take the values:

$$\begin{aligned}
&\mathrm{U}: \ u^* = 1.500 \qquad && f^* = 0 \\
&\mathrm{C1}: \ u^* = 0.181 \qquad && f^* = 0.158 \\
&\mathrm{C2}: \ u^* = 2.457 \qquad && f^* = 0.158\,.
\end{aligned}$$

4. Critical exponents

4.1. Asymptotic values

The values of critical exponents are determined by the values of the ζ-functions at the fixed point; the expressions for the ζ-functions related to the order parameter and the temperature field renormalization in the two-loop approximation read [9]:

$$\zeta_\psi = -3f + \frac{(n+2)}{72}u^2 + \frac{(11n+18)}{24}f^2 \tag{36}$$

$$\zeta_t = \frac{-(n+2)}{6}u + \frac{(n+2)}{12}u^2 - \frac{2(n+2)}{3}uf - \frac{(5n+1)}{2}f^2. \tag{37}$$

If there exists a stable fixed point, the critical exponent ν of the correlation length, the critical exponent γ of the order parameter susceptibility and the critical exponent α of the specific heat are given by

$$\nu = (2 - \zeta_\nu^*)^{-1} \tag{38}$$

$$\gamma = (2 - \zeta_\nu^*)^{-1}(2 - \zeta_\psi^*) \tag{39}$$

$$\alpha = (2 - \zeta_\nu^*)^{-1}(\varepsilon - 2\zeta_\nu^*) \tag{40}$$

where $\zeta_\nu = \zeta_\psi - \zeta_t$. From the analysis given above it follows that the charged fixed point C2 is the stable one and this results in values for the exponents (38)–(40) different from the values of the uncharged fixed point U, i.e. they are not given by the ^{4}He values as it is sometimes stated (see e.g. [1, 3, 7]). Trying to obtain their numerical values on the basis of the values of fixed point C2 coordinates $f^{*\mathrm{C,Padé}}$, $u^{*\mathrm{C2,Padé}}$ given in tables 1 and 2 in order to be self-consistent let us perform the same type of analytical continuation for the series for ζ-functions, as those which have been applied to the β-functions (4), (5). So introducing the auxiliary variable t let us represent functions (38)–(40) in the form of resolvent series in t and then we will chose the [1/1] Padé approximants for these series, which at $t = 1$ will give us the analytical continuation of the series requested. The expression obtained in such a way for a critical exponent ϕ ($\phi \equiv \{\nu, \gamma, \alpha\}$) reads

$$\phi = a_\phi^{(0)} \frac{1 + A_\phi}{1 + B_\phi} . \tag{41}$$

The expressions for the coefficients A_ϕ, B_ϕ in (41) read

$$A_\phi = a_\phi^{(1)} + B_\phi \qquad B_\phi = -a_\phi^{(2)}/a_\phi^{(1)} \tag{42}$$

and $a_\phi^{(i)}$ are to be determined from the resolvent series in t:

$$\phi = \sum_{i\geqslant 0} a_\phi^{(i)} t^i |_{t=1} . \tag{43}$$

Substituting (36) and (37) into (38)–(40) and representing (38)–(40) in the form of (43) one finds:

$$\begin{aligned} a_\nu^{(0)} &= \tfrac{1}{2} \\ a_\nu^{(1)} &= [(n+2)/12]\ u - \tfrac{3}{2} f \\ a_\nu^{(2)} &= [(n^2 - n - 6)/144]u^2 + [(71n + 138)/48] f^2 + [(n+2)/12]uf \end{aligned} \tag{44}$$

$$\begin{aligned} a_\gamma^{(0)} &= 1 \\ a_\gamma^{(1)} &= [(n+2)/12]u \\ a_\gamma^{(2)} &= [(n^2 - 2n - 8)/144]u^2 + [(5n+1)/4] f^2 + [5(n+2)/24]uf \end{aligned} \tag{45}$$

$$\begin{aligned} a_\alpha^{(0)} &= 1 \\ a_\alpha^{(1)} &= -[3(n+2)/12]u + \tfrac{9}{2} f \\ a_\alpha^{(2)} &= [(-3n^2 + 3n + 18)/144]u^2 - [(71n + 138)/16] f^2 - [(n+2)/4]uf . \end{aligned} \tag{46}$$

Considering the case $n = 2$ and substituting the coordinates of the fixed point C2 ($f^{*\mathrm{C,Padé}} = 0.158$), $u^{*\mathrm{C2,Padé}} = 2.457$ (see tables 1 and 2) into (44)–(46) one obtains for the critical exponents (38)–(40):

$$\nu = 0.857 \qquad \gamma = 1.880 \qquad \alpha = -1.141 . \tag{47}$$

The application of the Padé approximants for the analytical continuation of the functions may result in the appearance of poles in these functions. If the pole is located in a region of expansion parameters which is unphysical (e.g. negative coupling u or f) this does not complicate the analysis. This was the case for the β-functions in the region of couplings less than the fixed-point values. For the ζ-functions, however, considering the non-asymptotic behaviour (and thus being far from the stable fixed point) one passes through a region of couplings where the Padé approximation for the ζ-functions becomes ambiguous resulting in the appearance of a pole. Therefore studying the crossover behaviour in the next subsection we will still keep the polynomial representation for ζ-functions instead of the Padé approximants. Then for the asymptotic values of critical exponents one finds

$$\nu = 0.771 \qquad \gamma = 1.619 \qquad \alpha = -0.314 . \tag{48}$$

Comparing the values of (47) and (48) shows a numerical difference of 15% in ν and γ and a considerable increase in α. However, there is no qualitative change (e.g. the sign of the specific heat exponent remains the same).

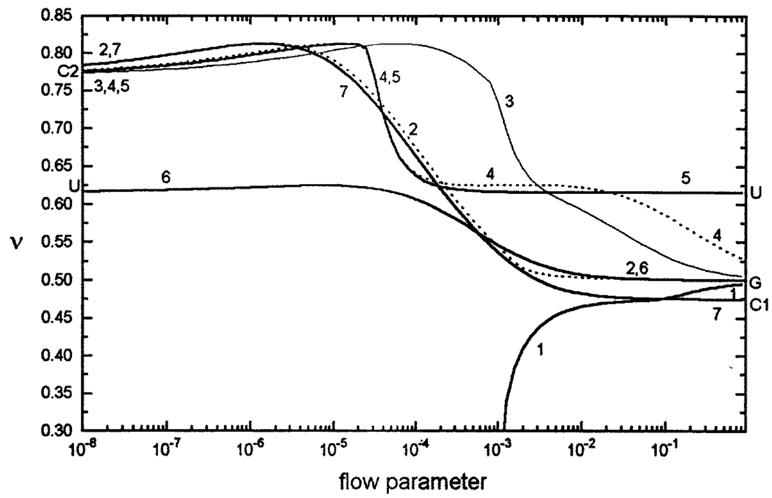

Figure 5. Effective exponent ν for the flows shown in figure 4 (for further description see the text).

4.2. Effective exponents and crossover

Effective exponents are usually defined by the logarithmic temperature derivatives of the corresponding correlation functions (see e.g. [37]). These can be found from the solutions of the renormalization-group equation for the renormalized vertex functions. These effective exponents contain two contributions, one from the corresponding ζ-functions now taken at the values $u(\ell)$, $f(\ell)$ of the flow curve considered ('exponent part'), and one from the change of the corresponding scaling function ('amplitude part'). The latter contributions will be neglected since we expect them to be smaller than the differences for the fixed point values of the exponents coming from the different treatments discussed before. Thus we have

$$\nu = (2 - \zeta_\nu(\ell))^{-1} \tag{49}$$

$$\gamma = (2 - \zeta_\nu(\ell))^{-1}(2 - \zeta_\psi(\ell)) \tag{50}$$

$$\alpha = (2 - \zeta_\nu(\ell))^{-1}(\varepsilon - 2\zeta_\nu(\ell))\,. \tag{51}$$

The flow parameter ℓ can be related to the relative temperature distance to T_c by the matching condition $t(\ell) = (\xi_0^{-1}\ell)^2$, with ξ_0 the amplitude of the correlation length.

We have computed these effective exponents, see figures 5–7, along the flow lines shown in figure 4 by inserting† the couplings $u(\ell)$ and $f(\ell)$ into (49)–(51). For the separatrix 1 we started with initial conditions leading to a flow, which did not stick in the fixed point C1 but slightly missed it, although the flow curve did not differ from the separatrix within the thickness of the lines shown in figure 4. For curve number 4 we started somewhat further away from the Gaussian fixed point G, leading to initial values of the effective exponents between their Gaussian values and their values for the uncharged fixed point U. Note that the values of the effective exponent γ for the uncharged fixed point U and the charged fixed point C1 are the same within the accuracy given by the scale of the figure.

† In fact, we have solved the flow equations (34), (35) starting near the unstable fixed points, for the initial value of the flow parameter we took $\ell = 1$. Using different initial values ($u(1)$ and $f(1)$ on the separatrix) would amount to rescaling the flow parameter.

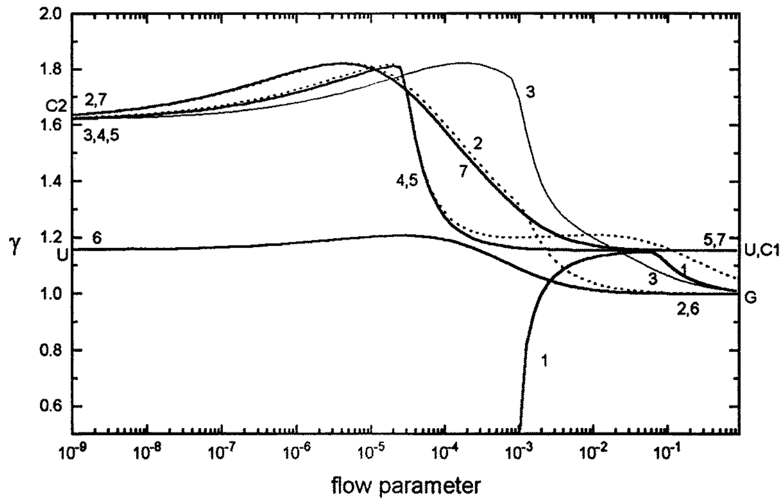

Figure 6. Effective exponent γ for the flows shown in figure 4 (for further description see the text).

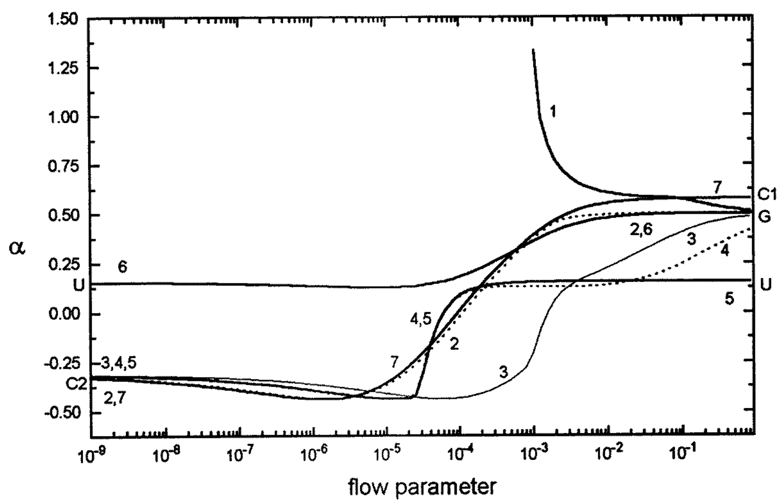

Figure 7. Effective exponent α for the flows shown in figure 4 (for further description see the text).

5. Conclusion

In the present paper we have re-examined expressions for the renormalization-group functions of the field-theoretical gauge model for a superconductor obtained previously in the two-loop approximation [9]. The main point which is discussed in this context is whether the equations for β-functions possess a stable fixed point or not. The absence of a stable fixed point is often interpreted as a change of the order of the phase transition (caused by the presence of magnetic field fluctuations) and evidence of the fluctuation-induced first-order phase transition. However, this change of the order of the phase transition (being of second order in the absence of a coupling to the gauge field) is confirmed only by perturbation theory calculations in low orders ([2], see [9] and references therein as well). By this paper we want to attract attention to the following problems:

(i) the series for the renormalization-group functions are not convergent and, generally speaking, it is desirable to first prove their asymptotic nature (as it is proven in the pure model case);

(ii) in the case where the series in the renormalized coupling constants are asymptotic and have zero radius of convergence, one should perform appropriate analytical continuation of these series in order to obtain reliable information on their basis.

In this paper we applied a simple Padé analysis to the series under discussion†. In the case of one coupling such an approach gives a qualitatively correct picture of the phase transition and restores the presence of a stable fixed point ([31], see formulae (16), (20) of this paper). The same situation happens here in the case of two couplings: at $n = 2$ 'uncharged' fixed point U (having coordinates $f^{*\mathrm{U,Padé}} = 0.158$, $u^{*\mathrm{U,Padé}} = 2.457$) appears to be stable, which leads to a new set of critical exponents. Of course, being calculated only in the two-loop approximation with application of Padé analysis, these values for the critical exponents are to be considered as preliminary ones. The main point we claim here is that in the frames of the renormalization-group analysis for the superconductor model there still exists the possibility of a second-order phase transition characterized by a set of critical exponents differing from those of ^{4}He.

Recently, the critical exponent η has been calculated in a self-consistent screening approximation [12] and a value of $\eta = -0.38$ was found. This may be compared with the values obtained for the charged fixed point C2 in our calculations using the asymptotic scaling law $\eta = 2 - \gamma/\nu$. From the values for γ and ν of (47) we find $\eta = -0.19$ and from (48) we find $\eta = -0.1$, respectively. These values for η are well within the physical range $\eta > -1$ at $d = 3$.

Acknowledgments

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† Because they are double series in two coupling constants, we made use of representing them in the form of resolvent series [35, 36] which enabled us then to pass to Padé analysis.

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